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Varieties and algebraic algebras of bounded degree

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Abstract

We study varieties of associative algebras over an infinite field. We prove that any variety is generated by an algebraic algebra of bounded degree over a commutative algebra and determine when it can be generated by an algebraic algebra over a field. We also give upper and lower bounds for the minimal algebraic degree of the algebra $M_n(G)$.

It is well known that an algebraic algebra of bounded degree n is a PI-algebra, satisfying, for example, the identity

$$\sum_{\sigma\in S(n+1)} (-1)^{\sigma} x^{\sigma(0)} y_1 x^{\sigma(1)} y_2 \dots y_n x^{\sigma(n)} = 0.$$

We prove the following theorem.

Theorem. *Any proper variety of associative algebras over a field of characteristic* 0 *can be generated by an algebraic algebra of bounded degree over some commutative algebra.*

Thus, for any PI-algebra A, there exists an algebraic algebra of bounded degree which has the same identities as A.

Let A be an associative algebra over a commutative ring R. An arbitrary R-linear mapping $Tr: A \to R$ will be called a trace. For example, in the full matrix algebras $M_n(F)$ over a field F, the mapping Tr can be defined by the formula

$$Tr\left(\sum_{i,j=1}^{n} a_{ij}e_{ij}\right) = \sum_{i=1}^{n} a_{ii}.$$

Note that we do not require that the trace satisfy the condition $Tr(a_1a_2) = Tr(a_2a_1)$.

Let X be a countable set, and let $F\langle X\rangle$ denote the free associative algebra (without unit) over a field F generated by the set X. Denote by T the free associative

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and commutative algebra with unit generated by all the elements $Tr(u)$, where u are nonempty words over X. We call the algebra $\widetilde{F}\langle X\rangle = F\langle X\rangle \otimes T$ the free algebra with trace. We will omit the symbol $\otimes$ below. The elements of the algebra $\widetilde{F}\langle X\rangle$ will be called polynomials with trace. Any polynomial with trace can be written as an F-linear combination of the monomials

$$u_0 Tr(u_1)\cdots Tr(u_n),$$

where $u_i \in F\langle X\rangle$.

Let A be an algebra with trace and $f(x_1,\dots,x_n)$ a polynomial with trace. We say the algebra A satisfies the identity with trace $f(x_1,\dots,x_n)=0$ if for any $a_1,\dots,a_n\in A$ the equality $f(a_1,\dots,a_n)=0$ is satisfied in A.

Polynomials with trace are said to have Cayley–Hamilton form of degree n if they have the form

$$x^n + \sum_{(i)} \alpha_{(i)} x^{i_0} Tr(x^{i_1})\cdots Tr(x^{i_t}),$$

where $(i)=(i_0,i_1,\dots,i_t)$ are various collections such that

$$i_1 \ge \cdots \ge i_t, \quad i_0+i_1+\cdots+i_t=n, \quad i_0<n.$$

Since $\operatorname{char} F=0$, the linearity of Tr implies by the usual multilinearization process that any trace identity is equivalent to a multilinear trace identity. The full matrix algebra of order n clearly satisfies the following identity with trace: $X_n(x)=0$, where $X_n(x)$ is a Cayley–Hamilton polynomial defined recursively by the formulas

$$X_1(x)=x-Tr(x), \qquad X_n(x)=X_{n-1}(x)\cdot x - \frac{1}{n}\cdot Tr(X_{n-1}(x)\cdot x).$$

Thus, the matrix algebra satisfies an identity of Cayley–Hamilton form. An arbitrary finite-dimensional algebra A can be embedded in algebra of matrices of finite order; therefore, a map $Tr: A\to F$ can be defined in A so that the algebra A satisfies some identity with trace of Cayley–Hamilton form.

Consider $A\otimes_F G$, where A is an arbitrary finite-dimensional algebra over a field F, $\operatorname{char} F=0$, and G is the Grassmann algebra of countable rank. Then $G=G_0\oplus G_1$, where G_0 (resp. G_1) is the subspace of G generated by the words of even (resp. odd) length. Since $C(G)=G_0$ is the centre of G, the algebra $A\otimes_F G$ can be considered as an algebra over the commutative algebra G_0.

If Tr is defined on the algebra A, then we define the G_0-linear mapping Tr in $A\otimes_F G$ as follows:

$$Tr(a\otimes g) = Tr(a)\cdot Tr(g), \quad a\in A, \ g\in G,$$

where $Tr(g) \stackrel{\text{def}}{=} Tr(g_0+g_1)=g_0$, $g\in G$, $g_0\in G_0$, $g_1\in G_1$.

Since $G_0 = C(G)$, the mapping Tr in the algebra $A \otimes_F G$ is defined correctly.

Theorem 1. *Any proper variety of associative algebras over a field of characteristic* 0 *can be generated by an algebraic algebra of bounded degree over some commutative algebra.*

Proof. Indeed, by the theorem of Kemer [1], any proper variety of associative algebras is generated by the Grassmann hull of some finite-dimensional superalgebra. Thus, it is sufficient to prove that Grassmann hull of an arbitrary finite-dimensional superalgebra satisfies some identity with trace of Cayley–Hamilton form. This implies that the Grassmann hull is an algebraic algebra of bounded degree over the commutative algebra G_0. We prove it in more explicit form.

Lemma 1. *If an algebra A satisfies multilinear identity of Cayley–Hamilton form of degree n, $\dim A = m$, then the algebra $A \otimes_F G$ satisfies some identity of Cayley–Hamilton form of degree $n(mn+1)$.*

Proof. Let $\{a_1, \dots, a_m\}$ be a basis of A, $\dim A = m$ and $f(x_1, \dots, x_n) = 0$, $\deg f = n$ be a multilinear identity with trace equivalent to a Cayley–Hamilton identity which is satisfied in A. Trivially, it is sufficient to prove that $A \otimes_F G$ satisfies the identity

$$(f(x_1, \dots, x_n))^{mn+1} = 0.$$

Consider arbitrary elements $b_1, \dots, b_n \in A \otimes_F G$,

$$b_l = \sum_{i=1}^{m} a_i \otimes g_{(i)}^{(l)}, \quad g_{(i)}^{(l)} \in G, \quad l = 1, \dots, n$$

and substitute them into the given identity with trace. Since the polynomial $f(x_1, \dots, x_n)$ is multilinear, we obtain

$$(f(b_1, \dots, b_n))^{mn+1} = \left(f\left(\sum_{i_1=1}^{m} a_{i_1} \otimes g_{(i_1)}^{(1)}, \dots, \sum_{i_n=1}^{m} a_{i_n} \otimes g_{(i_n)}^{(n)} \right) \right)^{mn+1}$$

$$= \left(\sum_{(i)} f\left(a_{i_1} \otimes g_{(i_1)}^{(1)}, \dots, a_{i_n} \otimes g_{(i_n)}^{(n)} \right) \right)^{mn+1}$$

$$= \left(\sum_{(i)} f\left(a_{i_1} \otimes \left(g0_{(i_1)}^{(1)} + g1_{(i_1)}^{(1)} \right), \dots, a_{i_n} \otimes \left(g0_{(i_n)}^{(n)} + g1_{(i_n)}^{(n)} \right) \right) \right)^{mn+1},$$

where $(i) = (i_1, \dots, i_n)$, $i_t \in \{1, \dots, m\}$. We notice $g0_{(i_t)}^{(t)} \in G_0$, $g1_{(i_t)}^{(t)} \in G_1$.

Divide the sum

$$\sum_{(i)} f\left(a_{i_1} \otimes \left(g0_{(i_1)}^{(1)} + g1_{(i_1)}^{(1)} \right), \dots, a_{i_n} \otimes \left(g0_{(i_n)}^{(n)} + g1_{(i_n)}^{(n)} \right) \right)$$

into two parts, where the first part consists of the summands of the form

$$f\left(a_{i_1}\otimes g_{0}{}^{(1)}_{(i_1)},\dots,a_{i_n}\otimes g_{0}{}^{(n)}_{(i_n)}\right)$$

and the second one consists of all the other summands.

Since $G_0=C(G)$ and the mapping Tr is G_0-linear the first part equals 0 in $A\otimes_F G$, because the algebra A satisfies the identity $f(x_1,\dots,x_n)=0$.

Note also that if $P_n(x_1,\dots,x_n)$ is an arbitrary multilinear polynomial with trace and $\rho_t\in\{0,1\}$ then

$$P_n\left(a_{i_1}\otimes g^{(1)}_{\rho_1},\dots,a_{i_n}\otimes g^{(n)}_{\rho_n}\right)=P'_n\left(a_{i_1},\dots,a_{i_n}\right)\otimes g^{(1)}_{\rho_1}\cdots g^{(n)}_{\rho_n},$$

where P'_n is some other multilinear polynomial with trace.

Denote $u_j=g^{(1)}_{\rho_1\,(i_1)}\cdots g^{(n)}_{\rho_n\,(i_n)}$, if there exist $t\in\{1,\dots,n\}$ such that $\rho_t=1$. Then we obtain

$$\left(\sum_{(i)}\left[f\left(a_{i_1}\otimes g_{0}{}^{(1)}_{(i_1)},\dots,a_{i_n}\otimes g_{1}{}^{(n)}_{(i_n)}\right)+\dots+f(a_{i_1}\otimes g_{1}{}^{(1)}_{(i_1)},\dots,a_{i_n}\otimes g_{1}{}^{(n)}_{(i_n)})\right]\right)^{mn+1}$$

$$=\left[\sum_{(i)}\sum_{j=1}^{2^n-1}f_j(a_{i_1},\dots,a_{i_n})\otimes u_j\right]^{mn+1}$$

$$=\sum_{(i),(j)}\left[\prod_{t=1}^{mn+1}f_{j_t}(a_{i_{1,t}},\dots,a_{i_{n,t}})\right]\otimes u_{j_1}\cdot u_{j_2}\cdots u_{j_{mn+1}}=0.$$

Here $(j)=(j_1,\dots,j_{mn+1})$, where $j_t\in\{1,\dots,2^n-1\}$, and

$$(i)=(i_{1,t},\dots,i_{n,t})\quad\text{where } i_{l,t}\in\{1,\dots,m\}.$$

Indeed, every word of the form u_{j_t} contains at least one element of the set $L=\{g_{1}{}^{(1)}_{(1)},\dots,g_{1}{}^{(1)}_{(m)},\dots,g_{1}{}^{(n)}_{(1)},\dots,g_{1}{}^{(n)}_{(m)}\}\subset G_1$. Since card $L=nm$, all words of the form $u_{j_1}\cdots u_{j_{mn+1}}$ contain two identical elements from L. Therefore, using $g_1^2=0$, where $g_1\in G_1$, obtain $u_{j_1}\cdot u_{j_2}\cdots u_{j_{mn+1}}=0$. Thus the lemma is proved. □

As observed above, an arbitrary finite-dimensional superalgebra A satisfies some multilinear identity with trace of Cayley–Hamilton form. Hence, by Lemma 1 the Grassmann hull of A also satisfies a Cayley–Hamilton form identity with trace. Hence the theorem is proved. □

We remark it is not true that any variety of associative algebras is generated by an algebraic algebra of bounded degree over a field.

Theorem 2. *A variety of associative algebras over a field of characteristic* 0 *is generated by an algebraic algebra of bounded degree over a field if and only if it can be generated by some finite-dimensional algebra.*

Proof. Let B be an algebraic algebra of degree n over a field F. Let N be the nil-radical of the algebra B. Then N satisfies the identity $x^n = 0$ and, by the Nagata–Higman theorem, N is nilpotent. The quotient algebra B/N has no nonzero nil-ideals, hence it satisfies the standard identity of some degree. It means the algebra B satisfies the identity

$$S_k(x_1,\dots,x_k)\cdot S_k(x_{k+1},\dots,x_{2k})\cdot\ \cdots\ \cdot S_k(x_{k(t-1)+1},\dots,x_{t\cdot k}) = 0.$$

The last identity obviously is not satisfied in the Grassmann algebra. Therefore, by the theorem of Kemer [1], $Var(B)$ can be generated by some finite-dimensional algebra C.

The converse assertion is trivial because a finite-dimensional algebra is always an algebraic algebra of bounded degree. Hence Theorem 2 is proved. □

Lemma 1 gives us an upper bound for the minimal degree d of the Cayley–Hamilton form identities of the algebra $M_n(G) = M_n(F)\otimes_F G : d \le n^4 + n$. Now we give the lower bound for this degree.

Theorem 3. *If the algebra $M_n(G)$ satisfies some multilinear identity with trace of Cayley–Hamilton form $P_d(x_1,\dots,x_d) = 0$ then $d \ge 2n$.*

Proof. Assume $\deg P_d < 2n$. Consider the matrix units $e_{11}, e_{12}, e_{22}, e_{23}, e_{33}, \dots, e_{n-1n}, e_{nn}$. There are $2n-1$ of them; therefore, we can choose of them the first d among them and make the substitution (below we assume $d = 2n-1$)

$$x_1 = e_{11}\otimes g_{11},\quad x_2 = e_{12}\otimes g_{12},\ x_3 = e_{22}\otimes g_{13},\dots, x_d = e_{nn}\otimes g_{1d}.$$

Here $g_{11}, g_{12},\dots, g_{1d}$ are arbitrary elements from G_1. But by the definition of the trace $Tr(a\otimes g_1) = Tr(a)\cdot Tr(g_1) = Tr(a)\cdot 0$, and $g_{1i_1}\cdot\ \cdots\ \cdot g_{1i_{2k+1}} \in G_1$, if $g_{1i_1},\dots,g_{1i_{2k+1}} \in G_1$. Hence, all monomials which are contained in P_d and have a subword of the form $Tr(x_{i_1}\cdot\ \cdots\ \cdot x_{i_{2k+1}})$ vanish. Consider the trace of even length $Tr(x_{l_1}\cdot\ \cdots\ \cdot x_{l_{2k}})$. After substitution we obtain

$$\begin{aligned} Tr(e_{i_1j_1}\otimes g_{1_1}\cdot\ \cdots\ \cdot e_{i_{2k}j_{2k}}\otimes g_{12k}) &= Tr(e_{i_1j_1}\cdot\ \cdots\ \cdot e_{i_{2k}j_{2k}}\otimes g_{11}\cdot\ \cdots\ \cdot g_{12k}) \\ &= Tr(e_{i_1j_1}\cdot\ \cdots\ \cdot e_{i_{2k}j_{2k}})\cdot g_{11}\cdot\ \cdots\ \cdot g_{12k} \\ &= 0\cdot g_{11}\cdot\ \cdots\ \cdot g_{12k} = 0. \end{aligned}$$

Since $e_{i_1j_1},\dots,e_{i_{2k}j_{2k}}$ are matrix units from the given set, either $e_{i_1j_1}\cdot\ \cdots\ \cdot e_{i_{2k}j_{2k}} = 0$, or $e_{i_1j_1}\cdot\ \cdots\ \cdot e_{i_{2k}j_{2k}} = e_{i_1j_{2k}}$, and since $2k > 1$, $i_1 < j_{2k}$ and we have $Tr(e_{i_1j_1}\cdot\ \cdots\ \cdot e_{i_{2k}j_{2k}}) = Tr(e_{i_1j_{2k}}) = 0$. Thus, any monomial containing the symbol Tr vanishes after substitution. It is evident that all the other monomials also vanish except $x_1\cdot x_2\cdot\ \cdots\ \cdot x_d$. Hence,

$$P_d(e_{11}\otimes g_{11},\dots,e_{dd}\otimes g_{1d}) = e_{11}\cdot\ \cdots\ \cdot e_{nn}\otimes g_{11}\cdot\ \cdots\ \cdot g_{1d} = e_{1n}\otimes g_{11}\cdot\ \cdots\ \cdot g_{1d} \neq 0.$$

Thus, any identity with trace of Cayley–Hamilton form of degree less than $2n$ is not satisfied in the algebra $M_n(G)$. Theorem 3 is thus proved. □

Consider an associative algebra B with unit over an infinite field of arbitrary characteristic. Let Γ be the ideal of identities of the algebra B. Let us prove the following lemma.

Lemma 2. *Let T be the T-ideal of $F\langle X\rangle$ generated by commutators. Consider any $f,g\in F\langle X\rangle$. If $f\notin T$, $g\notin\Gamma$, then $f\cdot g\notin\Gamma$.*

Proof. Consider the homogeneous in all variables components of $f\cdot g$. Since Γ is a T-ideal over infinite field it is enough to prove at least one of homogeneous components of $f\cdot g$ does not belong to Γ. Let us order the homogeneous components of an element of $F\langle X\rangle$ lexicographically on the degrees of the variables.

Denote by $\hat{f}$ a minimal homogeneous component of f which does not belong to T. It exists because $f\notin T$.

It is well known that any polynomial g can be written in the form

$$g=\sum_{i=1}^{s} h_i g_i, \tag{1}$$

where $h_i=x_{i_1}x_{i_2}\dots x_{i_m},\ i_1\le i_2\le\dots\le i_m$ and

$$g_i=\sum_{(j)}\alpha_{(j)}[x_{j_1},x_{j_2},\dots,x_{j_l}]\dots[x_{k_1},x_{k_2},\dots,x_{k_n}],$$

g_i are homogeneous in all variables.

Since $g\notin\Gamma$, there exist $g_i\notin\Gamma$ in the form (1). Let $\hat{g}$ be the minimal g_i satisfying this condition, and $\hat{h}$ is the minimal of multipliers h_i corresponding to $\hat{g}$ in (1).

Consider the homogeneous in all variables component of $f\cdot g$ of the same degrees as $\hat{f}\hat{h}\hat{g}$. It can be written in the form

$$\begin{aligned} q(x_{j_1},\dots,x_{m_v}) &= \hat{f}(x_{j_1},\dots,x_{j_r})\,\hat{h}(x_{l_1},\dots,x_{l_t})\hat{g}(x_{m_1},\dots,x_{m_v}) \\ &\quad +\sum_{i_1} f_{i_1}\cdot h_{i_1}\cdot\hat{g}+\sum_{i_2} f_{i_2}\cdot h_{i_2}\cdot g_{i_2} \pmod{\Gamma}, \end{aligned}$$

where f_{i_1}, f_{i_2} are homogeneous components of f and $h_{i_1}\hat{g}$, $h_{i_2}g_{i_2}$ are the components of g. The choice of $\hat{g}$ and $\hat{h}$ implies $g_{i_2}>\hat{g}$, $h_{i_1}>\hat{h}$ and because of homogeneity $f_{i_1}<\hat{f}$.

Let us change all variables of q the polynomials $\hat{f}\cdot\hat{h}$ and $\hat{g}$ depend on at the same time by z's and the other variables of $\hat{f}\cdot\hat{h}$ by y's with the same indices. Then

$$\begin{aligned} &q(y_{j_1},\dots,y_{l_{t'}},z_{j_{r'+1}},\dots,z_{l_t},x_{m_1},\dots,x_{m_{v'}}) \\ &= \hat{f}(y_{j_1},\dots,y_{j_{r'}},z_{j_{r'+1}},\dots,z_{j_r}) \\ &\quad\times\hat{h}(y_{l_1},\dots,y_{l_{t'}},z_{j_{t'+1}},\dots,z_{l_t})\hat{g}(x_{m_1},\dots,x_{m_{v'}},z_{m_{v'+1}},\dots,z_{m_v}) \\ &\quad+\sum_{i_1} f_{i_1}\cdot h_{i_1}\cdot\hat{g}+\sum_{i_2} f_{i_2}\cdot h_{i_2}\cdot g_{i_2} \pmod{\Gamma}. \end{aligned}$$

Consider the partial linearization of q on all variables z_l:

$$q(y_{j_1},\dots,z_l,\dots,x_{m_{v'}})|_{z_l=y_l-x_l} - q(y_{j_1},\dots,y_l,\dots,x_{m_{v'}}) - q(y_{j_1},\dots,x_l,\dots,x_{m_{v'}}).$$

Let $\tilde{q}$ be its homogeneous component with the same degrees in x's as degrees of $\hat{g}$. Obviously, it is enough to prove $\tilde{q}\notin\Gamma$.

Since $\deg_{x_m}\tilde{q}(y_{j_1},\dots,y_{l_t},x_{m_1},\dots,x_{m_v})=\deg_{x_m}\hat{g}(x_{m_1},\dots,x_{m_v})$ for any $m\in\{m_1,\dots,m_v\}$, modulo Γ $\tilde{q}$ is a sum of polynomials

$$\hat{f}(y_{j_1},\dots,y_{j_r})\hat{h}(y_{l_1},\dots,y_{l_t})\hat{g}(x_{m_1},\dots,x_{m_v}),$$

$$\sum\tilde{f}\cdot\tilde{h}\cdot\tilde{g},$$

where $\tilde{g}$ are partial linearizations of $\hat{g}$ depending on y's,

$$\sum\tilde{f}_{i_1}\cdot\tilde{h}_{i_1}\cdot\hat{g}(x_{m_1},\dots,x_{m_v}),$$

where $\tilde{f}_{i_1}$ are partial linearizations of $f_{i_1}<\hat{f}$ depending on y's only,

$$\sum\tilde{f}_{i_2}\cdot\tilde{h}_{i_2}\cdot\tilde{g}_{i_2},$$

where $\tilde{g}_{i_2}$ are partial linearizations of $g_{i_2}>\hat{g}$. Notice, in the last summand $\tilde{g}_{i_2}$ also depend on y's, since $\deg_{x_m}\sum\tilde{f}_{i_2}\cdot\tilde{h}_{i_2}\cdot\tilde{g}_{i_2}=\deg_{x_m}\hat{g}$ for any m and $g_{i_2}>\hat{g}$.

Let B be an F-algebra with unit such that $T[B]=\Gamma$. Since $\hat{g}\notin\Gamma$, there exist $b_1,\dots,b_v\in B$ such that $\hat{g}(b_1,\dots,b_v)\neq 0$. Consider the substitution $y_{j_1}=\cdots=y_{l_t}=1$, $x_{m_1}=b_1,\dots,x_{m_v}=b_v$ into $\tilde{q}$. Then the second and the fourth summands vanish, because $\tilde{g}$, $\tilde{g}_{i_2}$ are the sums of products of commutators. The third summand also vanishes; since $f_{i_1}<\hat{f}$, this implies $f_{i_1}\in T$ and $\tilde{f}_{i_1}(y_{n_1},\dots,y_{n_k})|_{y_j=1}=0$.

Thus

$$\tilde{q}(1,\dots,1,b_1,\dots,b_m)=\alpha\cdot\hat{g}(b_1,\dots,b_t)\neq 0,$$

where $\alpha\in F$, $\alpha=\hat{f}(1,\dots,1)\neq 0$, since $\hat{f}\notin T$. It means $\tilde{q}=0$ is not satisfied in B, therefore $\tilde{q}\notin\Gamma$. Thus the lemma is proved. □

We have shown above that if $\operatorname{char}F=0$ the variety $Var(G)$ cannot be generated by an algebraic algebra of bounded degree over a field. This is not true in the case where the field has nonzero characteristic.

Theorem 4. *If an associative algebra B with unit over infinite field of characteristic $p>0$ satisfies the Engel identity of some degree $[y,x,\dots,x]=0$, then $Var(B)$ can be generated by an algebraic algebra of bounded degree over some extension of the base field.*

Proof. We can assume that the field F is algebraically closed. Let Γ denote the ideal of identities of B. Let $\bar{F}\langle X\rangle=F\langle X\rangle/\Gamma$ be the relatively free algebra corresponding to Γ. $C(\bar{F}\langle X\rangle)$ denotes the centre of $\bar{F}\langle X\rangle$, and $J=J(\bar{F}\langle X\rangle)$ denotes the Jacobson radical of $\bar{F}\langle X\rangle$.

If B satisfies the Engel identity of some degree $[y,x,\dots,x]=0$, then for some n, $[x^{p^n},y]\in\Gamma$. Hence, $x^{p^n}\in C(\bar{F}\langle X\rangle)$ for any $x\in\bar{F}\langle X\rangle$. $Var(\bar{F}\langle X\rangle)$ is a nonmatrix variety; therefore, J is the verbal ideal generated by commutators. By the theorem of Kemer [2] J is nil of bounded degree; denote this degree by k.

Consider $S=\{x^{p^{n+l}}\mid x\in\bar{F}\langle X\rangle\}$, where $p^l\geq k$, and the subalgebra C generated by S. Then using $x^{p^n}\in C(\bar{F}\langle X\rangle)$, we have, for any $c\in C$,

$$c=\sum_{(i)}\alpha_{(i)}x_{i_1}^{p^{n+l}}\dots x_{i_s}^{p^{n+l}}=\sum_{(i)}\left(\beta_{(i)}x_{i_1}^{p^n}\dots x_{i_s}^{p^n}\right)^{p^l}=\left(\sum_{(i)}\beta_{(i)}x_{i_1}^{p^n}\dots x_{i_s}^{p^n}\right)^{p^l}.$$

Here, $x_{i_j}\in\bar{F}\langle X\rangle$, and $\alpha_{(i)}\in F$, $\beta_{(i)}$ is a solution of equation $y^{p^l}=\alpha_{(i)}$. Hence, for any $c\in C$ $c=x^{p^l}$. There are two cases: either $x\in J$ then $c=0$, or $x\notin J$ and c also does not belong to the commutator ideal. Therefore by Lemma 2 C is regular.

We construct using the central localization an algebra $A=\{fc^{-1}\mid f\in\bar{F}\langle X\rangle,\ c\in C\}$. Since C is regular and $C\subseteq C(\bar{F}\langle X\rangle)$ by the theorem of Rowen [3], $T[A]=\Gamma$. Let K be the field of fractions of C. Then $K\subseteq A$ and A is algebraic of degree p^{n+l} over the field K. Indeed, for any $a\in A$

$$a^{p^{n+l}}=(fc^{-1})^{p^{n+l}}=f^{p^{n+l}}(c^{p^{n+l}})^{-1}=k\quad\text{where}\quad k\in K,$$

because for any $f\in\bar{F}\langle X\rangle$, $f^{p^{n+l}}\in C$. Theorem 4 is thus proved. □

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